Characteristics, Bicharacteristics, and Geometric Singularities of Solutions of PDEs

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Lecture III: Bicharacteristics and the Hamilton-Jacobi Theory

Luca Vitagliano

University of Salerno, Italy

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Introduction

Let $\mathcal{E}$ be a PDE imposed on sections of a fiber bundle $E \longrightarrow M$. The wave-front of a singular solution is a characteristic surface. When the symbol of $\mathcal{E}$ does only depend on coordinates on M, the equation for characteristic surfaces is a first order *scalar PDE*. The theory of such PDEs is a part of *contact geometry*. Solutions come equipped with a 1-dimensional *bicharacteristic* foliation. Accordingly, *wave-fronts propagate along bicharacteristics*. Passing from the PDE for characteristic surfaces to the ODEs for bicharacteristics is a mathematical way to understand the passage from wave-optics to geometric optics, or from wave-mechanics to classical mechanics.

Plan of the Third Lecture

I will review the contact geometry of first order scalar PDEs with a special emphasis on bicharacteristics. In the case of a Hamilton-Jacobi equation the existence of bicharacteristic lines is stated by the *Hamilton-Jacobi Theorem*. I will briefly review the Hamilton-Jacobi theory. It relates the Hamilton-Jacobi equations for wave-fronts to the Hamilton equations for rays.

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1-Jets of Functions

In good cases, equations for characteristic surfaces are *first order PDEs in* 1 *dependent variable*, specifically, they are PDEs for *hypersurfaces in the space-time*.

For simplicity, I will consider *equations imposed on functions on a manifold* M, i.e., sections of the trivial bundle $M \times \mathbb{R} \longrightarrow M$. First jets are then denoted by J^1M.

Proposition

The Cartan distribution $\mathcal{C}$ *on* J^1M *is a* contact structure. *Moreover,* J^1M *possesses a canonical* contact form α.

α can be defined noting that

$$J^1M \longrightarrow T^*M \times \mathbb{R}, \quad (j^1f)(x) \longmapsto (d_xf, f(x)),$$

is a well defined diffemorphism. Thus, J^1M inherits a canonical function u and a canonical 1-form Θ. Put

$$\alpha := du - \Theta, \quad \text{locally, } \alpha = du - u_i dx^i.$$

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Remark

$d\alpha$ is non-degenerate on C, accordingly the tangent bundle splits as

$$TJ^1M = C \oplus \ker d\alpha$$

$\ker d\alpha$ possesses a canonical generator $\partial/\partial u$, the *Reeb field*, such that $\alpha(\partial/\partial u) = 1$. The cotangent bundle splits as

$$T^*J^1M = \mathrm{Ann}(\partial/\partial u) \oplus \langle \alpha \rangle.$$

Definition: Contact Fields

A *contact field* on J^1M is an infinitesimal symmetry of the Cartan distribution, i.e., a vector field X such that

$$[X, Y] \subset C, \quad \text{for all } Y \subset C,$$

or, equivalently,

$$L_X\alpha = \lambda\alpha, \quad \text{for some function } \lambda \text{ on } J^1M.$$

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Jacobi Algebra of the Canonical Contact Manifold

Functions on a simplectic manifold form a Poisson algebra. Similarly, *functions on a contact manifold (with a distinguished contact form) form a Jacobi algebra*.

Let $F \in C^\infty(J^1 M)$. Splits dF in two parts:

$$dF = F'\alpha + \omega_F, \quad \omega_F \in \mathrm{Ann}(\partial/\partial u).$$

Then $F' = \partial F/\partial u$.

Proposition

There is an $\mathbb{R}$-linear bijection $C^\infty(J^1 M) \longrightarrow \{\text{contact fields}\}$, given by

$$F \longmapsto X_F := F\,\partial/\partial u + Y_F, \quad \text{where } Y_F \subset C \text{ in defined by } -i_{Y_F} d\alpha = \omega_F.$$

Y_F is the *characteristic vector field of* F.

Then $[X_F, X_G] = X_{\{F,G\}}$, and $(M, \{-,-\})$ is a *Jacobi manifold*. Locally

$$X_F = D_i F \frac{\partial}{\partial u_i} - \frac{\partial F}{\partial u_i}\frac{\partial}{\partial x^i} + \left(F - u_i \frac{\partial F}{\partial u_i}\right)\frac{\partial}{\partial u}.$$

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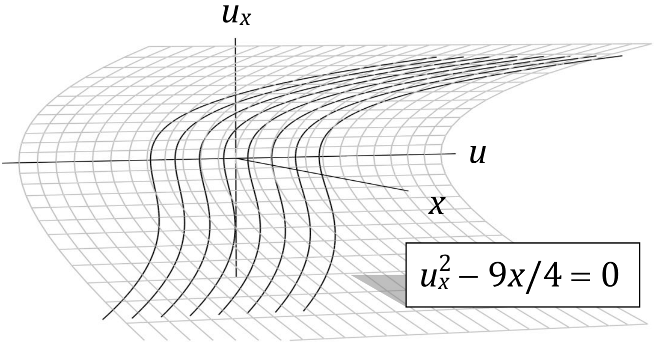
First Order Scalar PDEs and Bicharacteristics

A first order scalar PDE is an hypersurface $\mathcal{E} \subset J^1 M$. Thus, $\dim T\mathcal{E} = \dim C = 2n$, and $\dim C(\mathcal{E}) = T\mathcal{E} \cap C = 2n - 1$ at *generic points*.

Remark: Characteristic Foliation of $\mathcal{E}$

$d\alpha$ degenerates on $C(\mathcal{E})$ along a 1-dimensional distribution $\ell(\mathcal{E})$. Leaves of $\ell(\mathcal{E})$ are *characteristic lines of* $\mathcal{E}$. If $\mathcal{E}$ is the equation for characteristic surfaces of a PDE, then leaves of $\ell(\mathcal{E})$ are *bicharacteristics*.



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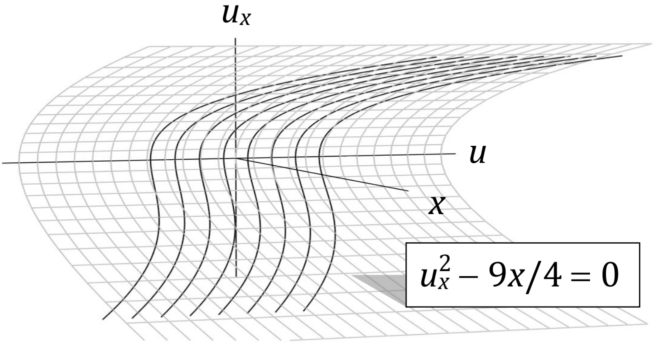

The Method of Characteristics

Theorem: Contact Hamilton-Jacobi

The characteristic distribution $\ell(\mathcal{E})$ is tangent to every solution of $\mathcal{E}$.

Remark: Initial Data for $\mathcal{E}$

An integral $(n-1)$-manifold $N \subset \mathcal{E}$ of $C(\mathcal{E})$ is naturally interpreted as a set of *Cauchy data for* $\mathcal{E}$. N corresponds to *non-characteristic Cauchy data* if it is everywhere transversal to $\ell(\mathcal{E})$.

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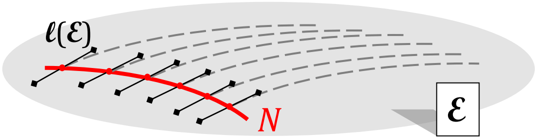
Let $N \subset \mathcal{E}$ be a set of non-characteristic Cauchy data for $\mathcal{E}$. Then, there exists a unique solution of the associated Cauchy problem, consisting in the union of characteristic lines through N.



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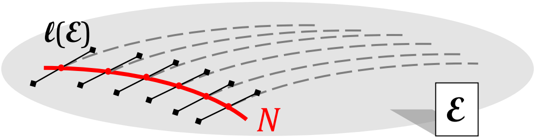

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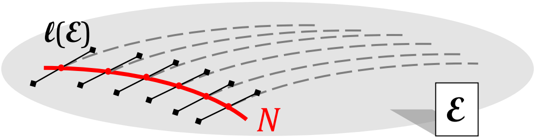

An Example

Remark

If $\mathcal{E} : F = 0$, then $\ell(\mathcal{E})$ is generated by the *characteristic vector field* $Y_F|_{\mathcal{E}}$. In this case, solving the Cauchy problem $(\mathcal{E}, N)$ amounts to solving the system of ODEs

$$\begin{cases} \dot{x}^i = -F_{u_i} \\ \dot{u}_i = F_{x^i} + u_i F_u \\ \dot{u} = -u_i F_{u_i} \end{cases}, \quad \text{with initial data on } N.$$

Example: $\mathcal{E} : u - u_x u_t = 0$ with Cauchy data $u|_{t=0} = x^2$

There exists a unique N encoding the Cauchy data:

$$N := \{(x = s,\ t = 0,\ u = s^2,\ u_x = 2s,\ u_t = s/2) : s \text{ a parameter}\}.$$

The *characteristic ODEs* are

$$\dot{x} = -u_t,\ \dot{t} = -u_x,\ \dot{u} = -2u_x u_t,\ \dot{u}_x = -u_x,\ \dot{u}_t = -u_t.$$

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Hamilton-Jacobi Equations

Let $\mathcal{E} : F = 0$ be a first order scalar PDE. In the case when F is independent of u, then *the Reeb vector field can be quotiented out* and $\mathcal{E}$ reduces to an hypersurface $\mathcal{H} : H = 0$ in T^*M, $H \in C^\infty(T^*M)$. $\mathcal{H}$ is then the *Hamilton-Jacobi equation* for the Hamiltonian system (T^*M, H).

Remark: Contact to Symplectic Dictionary

Hamilton-Jacobi equations can be treated as first order, scalar PDEs, applying the following "*Contact to Symplectic dictionary*":

Contact		Symplectic
contact manifold J^1M	$\longrightarrow$	symplectic manifold T^*M
First order scalar PDE, $F = 0$	$\longrightarrow$	Hamilton-Jacobi PDE, $H = 0$
First jet j^1f of $f \in C^\infty(M)$	$\longrightarrow$	differential df of f
Legendrian submanifold	$\longrightarrow$	Lagrangian submanifold
chracteristic vector field Y_F	$\longrightarrow$	Hamiltonian vector field X_H

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Hamilton-Jacobi Theorem

Let Ω be the symplectic structure on T^*M. Applying the *dictionary* one finds that *singular solutions of the Hamilton-Jacobi equation* $\mathcal{H} : H = 0$ *are Lagrangian submanifolds* N *of* T^*M *such that* $N \subset \mathcal{H}$.

Remark: Characteristic Foliation of $\mathcal{H}$

Ω degenerates on $\mathcal{H}$ along the 1-dimensional distribution $\ell(\mathcal{H})$ generated by the Hamiltonian vector field X_H.

Theorem: Symplectic Hamilton-Jacobi

The characteristic distribution $\ell(\mathcal{H})$ *is tangent to every solution of* $\mathcal{H}$.

Proposition: Symplectic Method of Characteristics

Let $N \subset \mathcal{H}$ *be an isotropic* $(n-1)$*-submanifold transversal to* $\ell(\mathcal{H})$*. Then, there exists a unique solution of* $\mathcal{H}$ *with initial data on* N*, consisting in the union of trajectories of* X_H *through* N.

One can get solutions of the Hamilton-Jacobi equation from solutions of the Hamilton equations!

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Complete Integrals of the Hamilton-Jacobi Problem

The converse is also true: one can get solutions of the Hamilton equations from solutions of the Hamilton-Jacobi equations!

Remark: Hamilton Jacobi Problem

The family of Hamilton-Jacobi equations $\mathcal{H}_E : H = E$ is referred to as the *Hamilton-Jacobi problem*. A *complete integral* is a Lagrangian foliation of T^*M whose leaves sit in the level surfaces of H.

A complete integral is the same as an n-parameter family of solutions depending on parameters $P_1, \ldots, P_n$ in an essential way.

Remark: Integrability of the Hamilton Equations

Applying the Hamilton-Jacobi theorem to the Hamilton-Jacobi equations $P_i = \text{const}$, one sees that $P_1, \ldots, P_n$ are n independent first integrals in involution: *the Hamiltonian system* (T^*M, H) *is integrable*.

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Generation of Canonical Transformations

Consider a complete integral of the Hamilton-Jacobi problem depending on parameters $P_1, \ldots, P_n$ varying in a parameter space P. Suppose that the leaves $L_{(P_1,\ldots,P_n)}$ are transversal to $T^*M \longrightarrow M$. Then, locally:

- there is an n-parameter family of functions $W_{(P_1,\ldots,P_n)} : M \longrightarrow \mathbb{R}$:
- $L_{(P_1,\ldots,P_n)}$ is the graph of $dW_{(P_1,\ldots,P_n)}$, and
- $\Phi := dW_{(-,\ldots,-)} : P \times M \longrightarrow T^*M$ is a diffeomorphism.

Remark

$P \times M$ inherits a symplectic structure $\Omega_W := \Phi^*(\Omega)$ and a Hamiltonian function $H_W := \Phi^*(H)$. The two Hamiltonian systems (T^*M, H) and $(P \times M, \Omega_W)$ are (locally) isomorphic. Moreover

- P_i and $Q^i := -\partial W/\partial P_i$ are conjugate: $\Omega_W = dP_i \wedge dQ^i$,
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The Hamilton equations on $P \times M$ are: $\dot{P}_i = 0$, $\dot{Q}^i = \text{const}$.

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Consider a complete integral of the Hamilton-Jacobi problem depending on parameters $P_1, \ldots, P_n$ varying in a parameter space P. Suppose that the leaves $L_{(P_1,\ldots,P_n)}$ are transversal to $T^*M \longrightarrow M$. Then, locally:

- there is an n-parameter family of functions $W_{(P_1,\ldots,P_n)} : M \longrightarrow \mathbb{R}$:
- $L_{(P_1,\ldots,P_n)}$ is the graph of $dW_{(P_1,\ldots,P_n)}$, and
- $\Phi := dW_{(-,\ldots,-)} : P \times M \longrightarrow T^*M$ is a diffeomorphism.

Remark

$P \times M$ inherits a symplectic structure $\Omega_W := \Phi^*(\Omega)$ and a Hamiltonian function $H_W := \Phi^*(H)$. The two Hamiltonian systems (T^*M, H) and $(P \times M, \Omega_W)$ are (locally) isomorphic. Moreover

- P_i and $Q^i := -\partial W/\partial P_i$ are conjugate: $\Omega_W = dP_i \wedge dQ^i$,
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Hamiltonian Dynamics of Boundary Data

Remark: A Canonical Solution to the Hamilton-Jacobi Problem

Suppose that the Hamiltonian system (T^*M, H) comes from a regular Lagrangian system, with Lagrangian $L \in C^\infty(TM)$. Then one can choose $P = M$ and *there is a canonical choice for* W:

$$W(x_0, x_1) = \int_{t_0}^{t_1} L(\gamma(t), \dot{\gamma}(t))dt$$

where γ is the solution of the Euler-Lagrange equations such that $\gamma(t_0) = x_0$, and $\gamma(t_1) = x_1$.

Remark: Hamiltonian Dynamics of Boundary Data

The diffeomorphism $\Phi^{-1} : T^*M \simeq M \times M$ "*transforms the symplectic manifold of initial data of Hamilton equations into a symplectic manifold of boundary data of the Euler-Lagrange equations*". In particular, *there is a Hamiltonian system on boundary data!*

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Conclusions

Consider a determined system of quasi-linear PDEs governing the dynamics of a field in the space-time. A wave-front is a *characteristic surface* in the space-time. In their turn, wave-fronts propagate along *bicharacteristics* which are often trajectories of a Hamiltonian system. I just described a mathematical way to pass from waves to rays, or from fields to particles (quantum wave-particle duality).

field equations $\Rightarrow$ characteristic surfaces $\Rightarrow$ bicharacteristics

is analogous to

quantum mechanics $\Rightarrow$ short wave-lenght $\Rightarrow$ classical mechanics.

In both transitions we progressively lose information: Performing the inverse transitions (quantizing?) requires additional information. A. M. Vinogradov conjectures that part of this information is contained in the singularity equations.

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References

Characteristics of First Order Scalar PDEs

- A. V. Bocharov et al., Symmetries and conservation laws for differential equations of mathematical physics, (I. S. Krasil'shchik, and A. M. Vinogradov eds.) *Transl. Math. Mon.* **182**, AMS, Providence - RI, 1999.

Geometric Hamilton-Jacobi Theory

- S. Benenti and W. M. Tulczyjew, The geometrical meaning and globalization of the Hamilton-Jacobi method, Lect. Notes Math. **836**, Springer, Berlin, 1980, p. 484.
- J. F. Cariñena, et al., Geometric Hamilton-Jacobi theory, *Int. J. Geom. Meth. Mod. Phys.* **3** (2006) 1417.
- G. Marmo, G. Morandi, and N. Mukunda, The Hamilton-Jacobi theory and the analogy between classical and quantum mechanics, *J. Geom. Mech.* **1** (2009) 317.
- A. M. Vinogradov, and B. A. Kupershmidt, The Structure of Hamiltonian Mechanics, *London Math. Soc. Lect. Notes Ser.* **60**, Cambridge Univ. Press, London, 1981, p. 173.

References

Generalizations of the Hamilton-Jacobi Theory

- J. F. Cariñena, et al., Geometric Hamilton-Jacobi theory for nonholonomic dynamical systems, *Int. J. Geom. Meth. Mod. Phys.* **7** (2010) 431.
- M. de Leon, D. Martín de Diego, M. Vaquero, A universal Hamilton-Jacobi theory, e-print: arXiv:1209.5351.
- M. de Leon, J. C. Marrero, D. Martín de Diego, A Geometric Hamilton-Jacobi Theory for Classical Field Theories; e-print: arXiv:0801.1181.
- M. de Leon, J. C. Marrero, and D. Martin de Diego, Linear almost Poisson structures and Hamilton-Jacobi equation, e-print: arXiv:0801.4358.
- M. de Leon, et al., Hamilton-Jacobi theory in k-symplectic field theory, *Int. J. Geom. Meth. Mod. Phys.* **7** (2010) 1491.
- M. de Leon, and S. Vilariño, Hamilton-Jacobi theory in k-cosymplectic field theory, e-print: arXiv:1304.3360.
- D. Iglesias-Ponte, M. de Leon, and D. Martin de Diego, Towards a Hamilton-Jacobi theory for nonholonomic mechanical systems, J. Phys. A: Math. Theor. **41** (2008) 015205.

References

Generalizations of the Hamilton-Jacobi Theory

- M. Leok, T. Ohsawa, and D. Sosa, Hamilton-Jacobi theory for degenerateLlagrangian systems with holonomic and nonholonomic constraints, *J. Math. Phys.* **53** (2012) 072905.
- M. Leok, and D. Sosa, Dirac structures and Hamilton-Jacobi theory for Lagrangian mechanics on Lie algebroids, *J. Geom. Mech.* **4** (2012) 421.
- T. Ohsawa, and A. M. Bloch, Nonholonomic Hamilton-Jacobi equation and Integrability, *J. Geom. Mech.* **1** (2009) 461.
- L. V., The Hamilton-Jacobi formalism for higher order field theory, *Int. J. Geom. Meth. Mod. Phys.* **7** (2010) 1413.
- L. V., Hamilton-Jacobi diffieties, *J. Geom. Phys.* **61** (2011) 1932.
- L. V., Geometric Hamilton-Jacobi field theory, *Int. J. Geom. Meth. Mod. Phys.* **9** (2012) 1260008.

References

Covariant Phase Space and Space of Boundary Data

- C. Crnković, and E. Witten, Covariant Description of Canonical Formalism in Geometrical Theories, in *Three Hundred Years of Gravitation*, S. W. Hawking and W. Israel (Eds.), Cambridge University Press, Cambridge, 1987, p. 676.
- G. Moreno, The geometry of the space of Cauchy data of nonlinear PDEs, *CEJM*, in press; e-print: arXiv:1207.6290.
- G. Moreno, and L. Vitagliano, Covariant space + time splittings of classical field theories, in preparation. Talk delivered by the second author at the "*Third Iberoamerican meeting on Geometry, Mechanics and Control*", Salamanca, September 03–07, 2012.
- C. Rovelli, Covariant Hamiltonian formalism for field theory: Hamilton-Jacobi equation on the space $\mathcal{G}$, *Lect. Notes Phys.* **633** (2003) 36.
- G. J. Zuckerman, Action Principles and Global Geometry, in *Mathematical Aspects of String Theory*, S. T. Yau (Ed.), World Scientific, Singapore, 1987, p. 259.

Thank you!