Characteristics, Bicharacteristics, and Geometric Singularities of Solutions of PDEs

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Lecture I: Characteristic Cauchy Data for PDEs

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Introduction to the Mini-Course

Many physical systems are described by partial differential equations (PDEs). Determinism requires the Cauchy problem to be well-defined. Even when this is the case for generic Cauchy data, there may exist *characteristic Cauchy data*. Characteristics of PDEs play an important role both in Mathematics and in Physics. They are related with:

- Cauchy problems,
- Intermediate integrals,
- Classification of PDEs,
- Singularities of solutions.

Plan of the Mini-course

I will review the theory of characteristics and bicharacteristics of PDEs, with a special emphasis on intrinsic aspects, i.e., those aspects which are invariant under general changes of coordinates. The relation between characteristics and bicharacteristics can be physically understood as the relation between *wave-optics* and *geometric optics*, or *wave-machanics* and *classical mechanics*.

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Outline of the Mini-Course

I) Characteristic Cauchy Data for PDEs
- Cauchy Problems
- Characteristic Data
- Examples

II) Singularities of Solutions of PDEs
- PDEs and Jet Spaces
- Singular Solutions
- Fold-Type Singularity Equation

III) Bicharacteristics and the Hamilton-Jacobi Theory
- Contact Geometry of Jets of Functions
- First Order Scalar PDEs
- Hamilton-Jacobi Theory

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Roughly, *characteristic Cauchy data* (meaning both a *Cauchy surface*, and *initial data* on it) are those for which the Cauchy problem is ill-defined: it may fail in existence, or uniqueness. The existence of characteristic Cauchy data is a rather general feature of field equations in classical field theory. Actually, the *boundary of a disturbance in the field*, i.e., a *wave-front*, is a characteristic surface. Thus *characteristic surfaces give relevant information about field wave propagation*.

Plan of the First Lecture

I will review the definition of characteristic Cauchy data, and the physical interpretation of a characteristic surface as a *wave-front*, in one of the simplest cases: *a determined system of quasi-linear PDEs*. I will also provide several examples from Mathematical Physics.

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Cauchy Problems in Normal Form

Consider a field propagating in the space-time:

- $\boldsymbol{u} = (u^1, \ldots, u^m)$ a vector of field variables,
- $x = (\mathbf{x}^1, \ldots, \mathbf{x}^{n-1}, t)$ split space + time coordinates,
- $\boldsymbol{u}_{\ell,J} := \partial^{\ell+|J|}\boldsymbol{u}/\partial t^\ell \partial \mathbf{x}^J$ space + time derivatives of the field.

Suppose that the field equations are in the following *normal form*:

$$\frac{\partial^k \boldsymbol{u}}{\partial t^k} = \boldsymbol{f}(t, \mathbf{x}, \ldots, \boldsymbol{u}_{\ell,J}, \ldots), \quad \ell < k, \ |J| + \ell \leq k. \quad (NF)$$

Theorem: Cauchy-Kowalewski

If the field equations (NF) are analytic*, they have a unique solution for any choice of* analytic *initial data*

$$\left.\frac{\partial^\ell \boldsymbol{u}}{\partial t^\ell}\right|_{t=t_0} = \boldsymbol{h}_\ell(\mathbf{x}), \quad \ell < k,$$

on the Cauchy surface $\Sigma : t = t_0$.

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Take a *covariant* point of view:

- $x = (x^1, \ldots, x^n)$ space-time coordinates,
- $u_I = \partial^{|I|} u / \partial x^I$ space-time derivatives of the field.

Suppose the field equations are k-th order, *determined*, and *quasi-linear*:

$$A^{i_1 \cdots i_k}(x, \ldots, u_J, \ldots) \cdot u_{i_1 \cdots i_k} = g(x, \ldots, u_J, \ldots), \quad |J| < k. \quad (QL)$$

Example: Euler-Lagrange Equations

Euler-Lagrange equations are always in the form (QL):

$$\frac{1}{I!J!} \frac{\partial^2 L}{\partial u_{i_1 \cdots i_\ell} \partial u_{j_1 \cdots j_\ell}} \cdot u_{i_1 \cdots i_\ell j_1 \cdots j_\ell} = g(x, \ldots, u_J, \ldots), \quad |J| < 2\ell$$

where $L = L(x, \ldots, u_I, \ldots)$, $|I| \leq \ell$, is the *Lagrangian density*.

Initial data can be assigned on a *generic* Cauchy surface $\Sigma : z(x) = 0$:

$$\left. \frac{\partial^\ell u}{\partial z^\ell} \right|_{z=0} = h_\ell, \quad \ell < k, \quad h_\ell \text{ functions on } \Sigma.$$

Determined, Quasi-linear Cauchy Problems

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Principal Symbol

The set of $m \times m$ square matrices $\{A^{i_1 \cdots i_k}\}$ is the *symbol* of the quasi-linear differential operator $A^{i_1 \cdots i_k} u_{i_1 \cdots i_k} - g$. *The symbol transforms like a symmetric contravariant tensor under a change of coordinates*:

$$x \longrightarrow \bar{x} = \bar{x}(x) \quad \Longrightarrow \quad A^{i_1 \cdots i_k} \longrightarrow \bar{A}^{i_1 \cdots i_k} = \frac{\partial \bar{x}^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial \bar{x}^{i_k}}{\partial x^{j_k}} A^{j_1 \cdots j_k}.$$

Under $x \longrightarrow (\mathbf{y}, z)$ the field equations become:

$$\frac{\partial z}{\partial x^{j_1}} \cdots \frac{\partial z}{\partial x^{j_k}} A^{j_1 \cdots j_k} \cdot \frac{\partial^k u}{\partial z^k} = \bar{f}(z, \mathbf{y}, \ldots, \bar{u}_{\ell, J}, \ldots), \quad \ell < k, \ |J| + \ell \leq k.$$

Remark: Non-Characteristic Cauchy Data

If the Cauchy data $(\Sigma, \{h_\ell\})$ are such that

$$\det \left(\frac{\partial z}{\partial x^{j_1}} \cdots \frac{\partial z}{\partial x^{j_k}} A^{j_1 \cdots j_k} \right) \Big|_{z=0} \neq 0,$$

the Cauchy problem can be recast in normal form around Σ.

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Characteristic Variety

We are led to consider the following *characteristic equation* for a covector $\boldsymbol{p} = (p_1, \ldots, p_n)$ on the space-time:

$$\det \boldsymbol{A}(\boldsymbol{p}) = 0, \quad \boldsymbol{A}(\boldsymbol{p}) := p_{i_1} \cdots p_{i_k} \boldsymbol{A}^{i_1 \cdots i_k}.$$

The characteristic equation defines a *characteristic variety* in the cotangent space of the space-time. Notice that *the characteristic variety may depend on a point in the "manifold of space-time derivatives of the field"*.

Characteristic covectors

Characteristic covectors, i.e., points of the characteristic variety, play an important role for:

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Characteristic Cauchy Data

Assume $A(p)$ is invertible for a generic p.

Definition: Characteristic Cauchy Data

The Cauchy data $(\Sigma, \{h_\ell\})$ are *characteristic* if

$$\det A(dz)|_{z=0} = 0, \quad (CE)$$

i.e., if $d_x z$ belongs to the characteristic variety for $x \in \Sigma$.

Equation (CE) is a PDE for the initial data $(\Sigma, \{h_\ell\})$. E.g., when $z = t - \tau(\mathbf{x})$, it becomes

$$\det \left(\sum_{s=0}^{k} \frac{\partial \tau}{\partial \mathbf{x}^{a_1}} \cdots \frac{\partial \tau}{\partial \mathbf{x}^{a_s}} B^{a_1 \cdots a_s} \right) = 0, \quad B^{a_1 \cdots a_s} := \tfrac{k!}{s!} A^{a_1 \cdots a_s n \cdots n}.$$

Remark

When the symbol does only depend on x, e.g. for *linear equations*, then (CE) is a scalar, first order PDE for the Cauchy surface Σ and can be treated, for instance, with the *method of characteristics*.

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Compatibility Conditions for Characteristic Initial Data

Remark: Characteristic Cauchy Problems I

The Cauchy problem may be ill-defined on a *characteristic surface*: the Cauchy-Kowalewski theorem may fail in existence. This is because *generic initial data on a characteristic surface are incompatible*.

In order to be *admissible*, initial data on a characteristic surface must fulfill suitable *compatibility conditions*. Indeed,

$$\operatorname{rank} A(dz)|_{z=0} < m \quad \Longrightarrow \quad (M \cdot A(dz))|_{z=0} = 0$$

for some maximal rank matrix $M = M(z, y, \ldots, \bar{u}_{\ell,J}, \ldots)$. Then

$$(M \cdot \bar{f})(z, y, \ldots, \bar{u}_{\ell,J}, \ldots)\big|_{z=0} = 0.$$

which can be interpreted as a PDE *constraining* the initial data

$$h_\ell = \left.\frac{\partial^\ell u}{\partial z^\ell}\right|_{z=0}.$$

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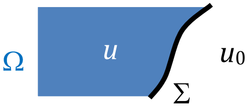
Remark: Characteristic Cauchy Problems II

On a *characteristic surface*, the Cauchy-Kowalewski theorem may fail in uniqueness, even for compatible, analytic initial data.

- u_0 a fiducial, smooth solution (the *background*),
- Ω a space-time region bounded by Σ.
- u a solution such that $u = u_0$ outside Ω, but $u \neq u_0$ in Ω.



Σ is the *wave-front* of the *wave* u propagating in the background u_0. Derivatives of u are continuous along Σ up to the order $k-1$, i.e., u and u_0 solve the same Cauchy problem $\Longrightarrow$ Σ is a characteristic surface.

- *Mathematically*: singularities occur along characteristic surfaces,
- *Physically*: wave-fronts of field waves are characteristic surfaces.

Physical Interpretation of Cauchy Surfaces

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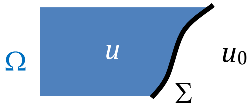

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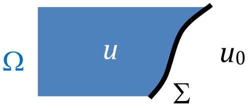

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Characteristic Cauchy Data for a Gauge Theory

Consider a Lagrangian field theory with Lagrangian density

$$L = L(x, \dots, \boldsymbol{u}_I, \dots), \quad |I| < \ell.$$

If L possesses gauge symmetries, then

$$\det \left(\frac{\partial^2 L}{\partial \boldsymbol{u}_I \partial \boldsymbol{u}_J} \right)_{|I|=|J|=\ell} = 0, \quad \text{identically.}$$

Then, for a *gauge theory* the *characteristic determinant* $\det A(\boldsymbol{p})$ vanishes identically. Assume $\operatorname{rank} A(\boldsymbol{p}) = r < m$ for a generic $\boldsymbol{p}$.

Definition: Characteristic Cauchy Data for Gauge Theories

The Cauchy data $(\Sigma, \{\boldsymbol{h}_\ell\})$ are *characteristic* if

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In this general case, characteristic Cauchy data still play a role in Cauchy problems and the theory of sigularity propagation.

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Characteristic Cauchy Data for Fully Non-Linear PDEs

Consider a *fully non-linear* determined system of k-th order PDEs:

$$\boldsymbol{F}(x, \ldots, \boldsymbol{u}_I, \ldots) = 0, \quad |I| \leq k.$$

A careful use of the *inverse function theorem* shows that the *analytic* Cauchy problem is well defined on a Cauchy surface $\Sigma : z = 0$ if

$$\det \left(\frac{\partial z}{\partial x^{j_1}} \cdots \frac{\partial z}{\partial x^{j_k}} \boldsymbol{A}^{j_1 \cdots j_k} \right) \Bigg|_{z=0} \neq 0, \quad \boldsymbol{A}^{j_1 \cdots j_k} := \frac{1}{I!} \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{u}_{j_1 \cdots j_k}}.$$

Remark: Characteristic Cauchy Data for Non-Linear PDEs

Characteristic Cauchy data can then be defined for non-linear PDEs exactly as for quasi-linear PDEs and play a similar role. In particular, *singularities of solutions of non-linear PDEs occur along characteristic surfaces.*

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$$\boldsymbol{F}(x, \ldots, \boldsymbol{u}_I, \ldots) = 0, \quad |I| \leq k.$$

A careful use of the *inverse function theorem* shows that the *analytic* Cauchy problem is well defined on a Cauchy surface $\Sigma : z = 0$ if

$$\det \left(\frac{\partial z}{\partial x^{j_1}} \cdots \frac{\partial z}{\partial x^{j_k}} \boldsymbol{A}^{j_1 \cdots j_k} \right) \Big|_{z=0} \neq 0, \quad \boldsymbol{A}^{j_1 \cdots j_k} := \frac{1}{I!} \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{u}_{j_1 \cdots j_k}}.$$

Remark: Characteristic Cauchy Data for Non-Linear PDEs

Characteristic Cauchy data can then be defined for non-linear PDEs exactly as for quasi-linear PDEs and play a similar role. In particular, *singularities of solutions of non-linear PDEs occur along characteristic surfaces*.

Example: Wave and Klein-Gordon Equations

Let $g = g_{ij} dx^i dx^j$ be a background (pseudo-)Riemannian metric on the "space-time". The field equation for the action

$$-\frac{1}{2}\int (g^{ij}\nabla_i u \nabla_j u - m^2 u^2)\sqrt{|\det g|}\, d^n x$$

is the linear equation

$$g^{ij}\nabla_i \nabla_j u + m^2 u = 0.$$

The characteristic variety is a quadric independent of m:

$$g^{-1}(p, p) = 0,$$

and characterictic surfaces $\Sigma : z = 0$ are defined by

$$g^{-1}(dz, dz)|_{z=0} = 0.$$

Characteristic Surfaces of Wave and Klein-Gordon Equations

- If g is Riemannian, *there are no characteristic surfaces*,
- if g is Lorentzian, *characteristic surfaces are null hypersurfaces*.

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Example: Dirac Equation

Let $\eta = \eta_{\mu\nu} dx^\mu dx^\nu$ be the Minkowski metric on the flat space-time. The Dirac equation for a 4-component spinor $u = (u^0, \ldots, u^3)$ is

$$(i\gamma^\mu \partial_\mu - m)u = 0,$$

where $\gamma^0, \ldots, \gamma^3$ are 4×4 Dirac matrices. The characteristic variety is independent of m:

$$\det A(p) = \det(i\gamma^\mu p_\mu) = (\eta^{-1}(p,p))^2 = 0,$$

and characteristic surfaces $\Sigma : z = 0$ are defined by

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Characteristic Surfaces for the Dirac Equation

Characteristic surfaces of the Dirac equation are null hypersurfaces ([Racah 32] interprets this result in terms of the Heisenberg principle).

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Let $g = g_{ij}dx^i dx^j$ be a background Lorentzian metric on a 4-dimensional space-time. The field equations for the action

$$-\int g^{ik}g^{j\ell}\nabla_{[i}u_{j]}\nabla_{[k}u_{\ell]}\sqrt{|\det g|}d^4x.$$

are the (vacuum) Maxwell equations:

$$g^{ik}\nabla_k(\nabla_i u_j - \nabla_j u_i) = 0,$$

where $u = (u_0, \ldots, u_3)$ is the potential. The characteristic equation

$$\det A(p) = \det(g^{-1}(p,p)\,\mathrm{I} - p^\sharp \otimes p) = 0$$

is satisfied identically. Indeed, $\operatorname{rank} A(p)$ is generically 3 and characteristic surfaces $\Sigma : z = 0$ are defined by $\operatorname{rank} A(dz)|_{z=0} < 3$.

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$\operatorname{rank} A(p) < 3$ iff p is a null covector $\Longrightarrow$ *characteristic surfaces of Maxwell equations* are null hypersurfaces.

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Example: Einstein Equations

Let $\boldsymbol{u} = u_{ij}dx^i dx^j$ be an unknown Lorentzian metric on a 4-dimensional space-time. The field equations for the action

$$\int u^{ij} R_{ij}[\boldsymbol{u}] \sqrt{|\det \boldsymbol{u}|} d^4 x$$

are vacuum Einstein equations:

$$\mathbf{Ric}[\boldsymbol{u}] = 0.$$

The characteristic equation

$$\det A(\boldsymbol{p}) = \det \left((2\delta_i^{[m} u^{k][\ell} \delta_j^{n]} - u^{n[m} u^{k]\ell} g_{ij}) p_k p_\ell \right) = 0$$

is satisfied identically. Indeed $\operatorname{rank} A(\boldsymbol{p})$ is generically 6 and characteristic surfaces are defined by $\operatorname{rank} A(dz)|_{z=0} < 6$.

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$\operatorname{rank} A(\boldsymbol{p}) < 6$ iff $\boldsymbol{u}^{-1}(\boldsymbol{p}, \boldsymbol{p}) = 0 \Longrightarrow$ *characteristic surfaces are null hypersurfaces w.r.t.* $\boldsymbol{u}$. In fact, they depend on initial data.

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Example: A Fully Non-Linear (Unphysical) Example

Consider the scalar PDE in two independent variables x, t:

$$u_{xxx} - (u_{ttx})^2 + u_{ttt}u_{txx} = 0.$$

For $\boldsymbol{p} = pdx + qdt$

$$\boldsymbol{A}(\boldsymbol{p}) = u_{txx}q^3 - 2u_{ttx}q^2q + u_{ttt}qp^2 + p^3,$$

which is generically non-zero. Accordingly, a hypersurface $\Sigma : z(x,t) = 0$ is characteristic iff

$$(u_{txx}z_t^3 - 2u_{ttx}z_t^2z_x + u_{ttt}z_tz_x^2 + z_x^3)|_{z=0} = 0.$$

Since $z_t \neq 0$, one can search for Σ in the form $\Sigma : t = \tau(x)$ and get

$$u_{txx} + 2u_{ttx}\tau_x + u_{ttt}\tau_x^2 - \tau_x^3 = 0$$

which depends on initial data on Σ.

Conclusions

Characteristic Cauchy data for a PDE are basically those for which the Cauchy problem is ill-defined. Characteristic surfaces are Cauchy surfaces of Caharacteristic Cauchy data. They have nice mathematical and physical interpretations as *hypersurfaces along which solutions may exhibit singularities*, and *wave-fronts of field waves*. Characteristic surfaces of Klein-Gordon, Dirac, Maxwell, and Einstein equations are null hypersurfaces in the space-time, which *physically* corresponds to the fact that the *phase velocity* of the corresponding fields is the *speed of light*.

So far, I worked in local coordinates, but most of what I discussed is independent of coordinates. In the next lecture, I will present a differential geometric theory of characteristics and singularities of solution of PDEs which is manifestly coordinate independent. Jet spaces will play a major role.

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